Applied Numerical Mathematics 69 (2013) 59–72

Contents lists available at SciVerse ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

CrossMark

[b] Sch f Ma he a ica Scie ce , Pe i g U i e i , Beiji g 100871, Pe e' Re b ic f Chi a

ARTICLE INFO

A ic e hi :
Received 27 February 2012
Received in revised form 4 March 2013
Accepted 13 March 2013
Available online 21 March 2013

Ke d :
Spectral approximation
Multiple eigenvalue
Finite element
Asymptotic expansion
Splitting extrapolation

ABSTRACT

This paper discusses the extrapolation of numerical eigenvalues by finite elements for differential operators and obtains the following new results: (a) By extending a theorem of eigenvalue error estimate, which was established by Osborn, a new expansion of eigenvalue error is obtained. Many achievements, which are about the asymptotic expansions of finite element methods of differential operator eigenvalue problems, are brought into the framework of functional analysis. (b) The Richardson extrapolation of nonconforming finite elements for multiple eigenvalues and splitting extrapolation of finite elements based on domain decomposition of non-selfadjoint differential operators for multiple eigenvalues are achieved. In addition, numerical examples are provided to support the theoretical analysis.

1. Introduction

The Richardson extrapolation is a well-known technique to construct high-order methods in numerical analysis. It is applicable to many problems, including ordinary or partial differential equations. All these applications are based on the existence of an error expansion for the discrete approximations in a single mesh parameter (see [16]). To produce more accurate approximations for partial differential equations, in the last 30 years, many scholars studied the Richardson extrapolation of finite element methods, e.g. see [1,2,4,9,10,13,24,25,28,33] and the references therein.

As for multidimensional problems, the Richardson extrapolation is costly since it considers just a single parameter. So, the splitting extrapolation, which is based on multivariate expansions with several mesh parameters, appears. Since 1980s, the splitting extrapolation has been developed widely in the numerical analysis community. The splitting extrapolation is a better technique to deal with the so-called curse of dimensionality and is also a highly parallel algorithm (see [20] and the book review [31]). It is especially important that the splitting extrapolation is also applied to the finite element methods, see [7,14,19,27,28], etc.

During the development of the extrapolation of finite element methods, the extrapolation for eigenvalue problems is an attractive issue, e.g. see [5,15,18,19,21,22,24,25,27,28,34]. Especially, [5] studied successfully the extrapolation of conforming finite elements for multiple eigenvalues of selfadjoint differential operator. However, to the best of our knowledge, there has no research on nonconforming finite elements extrapolations for multiple eigenvalues and the finite element extrapolations for multiple eigenvalues of non-selfadjoint differential operator. [22,24] discussed the extrapolation of nonconforming finite element eigenvalues, e.g., the asymptotic expansion of the EQ_1 element

☆ Supported by the National Natural Science Foundation of China (Grant No. 11161012) and Science and Technology Fundation of GuiZhou Province of China (No. [2011]2111).
* Corresponding author.
E- ai add e e : ydyang@gznu.edu.cn (Y.D. Yang), bihaimath@gznu.edu.cn (H. Bi), srli@pku.edu.cn (S. Li).

0168-9274/$36.00
http://dx.doi.org/10.1016/j.apnum.2013.03.002

$$\lambda_h - \lambda = -\frac{h_1^2 + h_2^2}{3} \int_{\Omega} (\partial_1 \partial_2 \ )^2 + \vartheta(h^4) \tag{1.1}$$

was proved where is the limit of $_h$, the eigenfunction corresponding to λ_h (see Theorem 3.1 in [22]). It is an important work. By using this expansion one can extrapolate the simple eigenvalue to obtain high-precision eigenvalues. However, in the case of multiple eigenvalues, from the spectral approximation theory we know that when h changes, the exact eigenfunction which the finite element eigenfunction $_h$ approximates to also changes (e.g., see Theorem 7.4 in [3]), thus, is related to h in the expansion. So it cannot be guaranteed that we use this expansion directly to extrapolate and get high-precision eigenvalues. As for the extrapolation of finite elements for non-selfadjoint differential operator eigenvalue problems, including the splitting extrapolation, [27] and [28] proved the asymptotic expansion for simple eigenvalues (see Theorem 2 in [27])

$$\lambda_h - \lambda = \sum_{i=1} \beta_i(\ )\bar{h}_i^2 + \vartheta(h_0^4), \tag{1.2}$$

where $\beta_i(\ )$ is independent of h. By this expansion the splitting extrapolation for simple eigenvalues can be achieved. But for multiple eigenvalues, $\beta_i(\ )$ is related to h in (1.2). When the ascent of λ is larger than 1, not only is $\beta_i(\ )$ related to h but also the accuracy of λ_h relates to the ascent which was pointed out in [28]. So we cannot use this expansion directly to get high-precision eigenvalues. The simple eigenvalue is a strong condition since the eigenvalue of non-selfadjoint problems is not simple in general and its ascent is probably larger than 1. This paper aims to study the extrapolation of finite elements for multiple eigenvalues including the case that the ascent is equal to or larger than 1.

We develop the previous corresponding investigations and obtain the main results which are in Sections 3 and 4 in this paper. Special works of this paper are as follows:

(a) In Section 3, we provide an eigenvalue error expansion (see (3.1)). This expansion is a simple extension of the estimate which was established by Osborn (see (2.4) in this paper). In many applications the first term on the right hand side of (3.1)/(2.4) is the dominant term and the second term is of higher order than the first one. The error is determined by the first term. Compared with (2.4), the advantage of (3.1) is that it indicates that the dominant term is effectively the size of the error. It is this feature that leads to the asymptotic formula for the error. Thus we bring the extrapolation of finite elements for differential operator eigenvalue problems into the framework of functional analysis.

(b) In Section 4, the asymptotic expansion of finite elements for differential operator eigenvalue problems is discussed by using Theorem 3.1. In Section 4.1 we give and prove the asymptotic expansions of EQ_1 element for multiple eigenvalues. In Section 4.2, for second-order non-selfadjoint differential operator eigenvalue problems, the splitting extrapolation based on domain decomposition is discussed. We throw off the assumption that λ is a simple eigenvalue in Theorem 2 in [27] and realize the splitting extrapolation of finite elements for multiple eigenvalues.

Besides, in Section 5, some numerical experiments are reported to support our theory.

In this paper, C denotes a positive constant independent of h, which may stand for different values at its different occurrences.

2. Preliminaries

Let X be a separable complex Banach space with norm $\|\cdot\|$ and conjugate pairs $\langle\cdot,\cdot\rangle$, respectively. In this section, let $T: X \to X$ be a nonzero compact linear operator, $T_h: X \to X$ and $\{T_h\}_{h>0}$ be a family of compact operators, and $\|T_h - T\| \to 0$ $(h \to 0)$. Consider the following eigenvalue problem:

$$T \ = \mu \ , \tag{2.1}$$

and its approximation

$$T_h \ _h = \mu_h \ _h. \tag{2.2}$$

We use the eigenpairs of (2.2) to approximate to those of (2.1).

[12] has proved the following Lemma 2.1.

Lemma 2.1. *Le* $\{\mu_j\}$ *be a e e a i , f he , , e eige a e f T, each i e acc di g i i ici . The he e e i , a e e a i ,* $\{\mu_{j,h}\}$ *f he , , e eige a e f* T_h*, i h e e i i , , acc di g i i ici , , ch ha*

$$\mu_{j,h} \to \mu_j \quad (h \to 0), \ j = 1, 2, \ldots. \tag{2.3}$$

Set $\lambda_j = \frac{1}{\mu_j}$, $\lambda_{j,h} = \frac{1}{\mu_{j,h}}$. In some papers μ_j and $\mu_{j,h}$ are called eigenvalues, and λ_j and $\lambda_{j,h}$ are called characteristic values. In our paper all of these are called eigenvalues.

Let μ be the -th eigenvalue of (2.1) with algebraic multiplicity , $\mu = \mu_{+1} = \cdots = \mu_{+-1}$. Then μ is also the eigenvalue of T' with algebraic multiplicity , where T' is the Banach adjoint of T. Let $\mu_{,h}$ be the -th eigenvalue of (2.2) and set $\hat{\mu}_{,h} = \frac{1}{}\sum_{j=}^{+-1}\mu_{j,h}$. Denote by $\rho(T)$ the resolvent set of T, and $\sigma(T)$ the spectrum of T.

Let Γ be a closed Jordan curve enclosing μ and Δ be a domain enclosed by Γ, $\Delta\backslash\mu \subset \rho(T)$. Let h be sufficiently small, then $\Gamma \subset \rho(T_h)$. The definitions of spectral projection and the ascent are as follows (see [3,8,29]):

Spectral projection. Denote $R(T) = (- T)^{-1}$, $R(T_h) = (- T_h)^{-1}$, define

$$E = E(\mu) = \frac{-1}{2i\pi}\int_{\Gamma} R(T)\,d ,$$

$$E_h = E_h(\mu) = \frac{-1}{2i\pi}\int_{\Gamma} R(T_h)\,d .$$

We call E the spectral projection associated with T and μ, and E_h the spectral projection associated with T_h and the eigenvalues of T_h which converge to μ. Let T'_h be the Banach adjoint of T_h. Similarly, we can define the spectral projection E' associated with T' and μ, and E'_h associated with T'_h and the eigenvalues of T'_h which converge to μ.

Ascent, generalized eigenvector. There exists the smallest integer α, called the ascent of $\mu - T$, such that the null space $e((\mu - T)^{\alpha}) = e((\mu - T)^{\alpha+1})$. The vectors in $e((\mu - T)^{\alpha})$ are called generalized eigenvectors of T corresponding to μ. Likewise, the ascent and generalized eigenvectors of $\mu_{,h} - T_h$, $\mu - T'$ and $\mu_{,h} - T'_h$ can be defined.

We denote $R(E)$, $R(E_h)$, $R(E')$ and $R(E'_h)$ as the image spaces of E, E_h, E' and E'_h, respectively. Then $R(E$

$$= \left\| \frac{-1}{2i\pi} \lim_{d(\pi)\to 0} \sum_{=1} R_{\xi} \ (T_h)(T - T_h) R_{\xi} \ (T) \Delta \quad f \right\|$$

$$\leqslant \frac{1}{2\pi} \ e \ g \ h(\Gamma) \sup_{\in \Gamma} \left\| R \ (T_h) \right\| \left\| (T - T_h)|_{R(E)} \right\| \sup_{\in \Gamma} \left\| R \ (T) \right\| \|f\|$$

$$\leqslant C \left\| (T - T_h)|_{R(E)} \right\| \|f\|.$$

Thus we get

$$\left\| (E_h - E)|_{R(E)} \right\| \leqslant C \left\| (T - T_h)|_{R(E)} \right\| \to 0 \quad (h \to 0). \tag{3.3}$$

For any $f \in R(E)$, $E_h f = 0$ implies

$$\|f\| = \|Ef - E_h f\| \leqslant \left\| (E_h - E)|_{R(E)} \right\| \|f\| \to 0 \quad (h \to 0),$$

i.e., $f = 0$. So $E_h|_{R(E)} : R(E) \to R(E_h)$ is one-to-one. Since $di \quad R(E_h) = di \quad R(E) = \ $, $E_h|_{R(E)}$ is onto. Hence $(E_h|_{R(E)})^{-1}$ exists and is defined on $R(E_h)$, and we write E_h^{-1} for $(E_h|_{R(E)})^{-1}$ for simplicity. For sufficiently small h and $f \in R(E)$ with $\|f\| = 1$, from (3.3) we have

$$1 - \|E_h f\| = \|Ef\| - \|E_h f\| \leqslant \left\| (E - E_h)|_{R(E)} \right\| \leqslant \frac{1}{2}.$$

Hence $\|E_h f\| \geqslant \frac{1}{2}\|f\|$. This implies E_h^{-1} is bounded, and for sufficiently small h, E_h^{-1} is uniformly bounded in h. Define

$$\hat{T}_h = E_h^{-1} T_h E_h|_{R(E)}. \tag{3.4}$$

Clearly,

$$\hat{T}_h : R(E) \to R(E).$$

Since $R(E_h)$ is the invariant subspace of T_h, $E_h E_h^{-1}$ is the identical operator on $R(E_h)$ and $E_h^{-1} E_h$ is the identical operator on $R(E)$, thus we have $\sigma(\hat{T}_h) = \{\mu_{\ ,h}, \ldots, \mu_{\ + \ -1,h}\}$. And we can see that the algebraic and geometric multiplicity of any $\mu_{i,h}$ as an eigenvalue of $\hat{T}_h$ is equal to its algebraic and geometric multiplicity as an eigenvalue of T_h. Write $\hat{T} = T|_{R(E)}$, by the spectral decomposition theorem we have $\sigma(\hat{T}) = \{\mu \ , \ldots, \mu_{\ + \ -1}\}$. Thus the traces of $\hat{T}$ and $\hat{T}_h$ can be obtained as follows, respectively,

$$ace\,\hat{T} = \quad \mu$$

$$\begin{aligned}\big\langle(\hat{T}-\hat{T}_h)\varphi_j,\varphi_j'\big\rangle&=\big\langle T\varphi_j-E_h^{-1}T_hE_h\varphi_j,\varphi_j'\big\rangle\\&=\big\langle E_h^{-1}E_h(T-T_h)\varphi_j,\varphi_j'\big\rangle\\&=\big\langle(T-T_h)\varphi_j,\varphi_j'\big\rangle+\big\langle\big(E_h^{-1}E_h-I\big)(T-T_h)\varphi_j,\varphi_j'\big\rangle.\end{aligned}\tag{3.6}$$

Let $L_h=E_h^{-1}E_h$, and it can be easily proved that L_h is the projection operator from X to $R(E)$ along $e\,(E_h)$. Hence L_h' is the projection operator from X' to $e\,(E_h)^{\perp}=R(E_h')$ along $R(E)^{\perp}=\ e\,(E')$. Thus

$$\big\langle(L_h-I)(T-T_h)\varphi_j,E_h'\varphi_j'\big\rangle=\big\langle(T-T_h)\varphi_j,\big(L_h'-I\big)E_h'\varphi_j'\big\rangle=0,$$

noting that $E'\varphi_j'=\varphi_j'$, we get

$$\big\langle(L_h-I)(T-T_h)\varphi_j,\varphi_j'\big\rangle=\big\langle(L_h-I)(T-T_h)\varphi_j,\big(E'-E_h'\big)\varphi_j'\big\rangle.\tag{3.7}$$

From (3.7), the boundedness of L_h, and (3.3) (applied to T' and T_h') we have

$$\begin{aligned}\big|\big\langle(L_h-I)(T-T_h)\varphi_j,\varphi_j'\big\rangle\big|&\leqslant\Big(\sup_h\|L_h-I\|\Big)\big\|(T-T_h)|_{R(E)}\big\|\,\big\|\big(E'-E_h'\big)\big|_{R(E')}\big\|\,\|\varphi_j\|\cdot\big\|\varphi_j'\big\|\\&\leqslant C\big\|(T-T_h)|_{R(E)}\big\|\,\big\|\big(T'-T_h'\big)\big|_{R(E')}\big\|.\end{aligned}\tag{3.8}$$

Finally, denote $R=\frac{1}{\ }\sum_{j=}^{+\ -1}\langle(L_h-I)(T-T_h)\varphi_j,\varphi_j'\rangle$; by using (3.5), (3.6) and (3.8), we obtain the desired result. □

Remark 3.1. Note that in (3.1) $\{\varphi_j\}^{\ +\ -1}$ is a basis of $R(E)$ and $\{\varphi_j'\}^{\ +\ -1}$ is the dual basis in $R(E')$, and it can be seen from the proof that they are all independent of h. Therefore, based on Theorem 3.1, we can realize the Richardson extrapolation of multiple eigenvalues and the splitting extrapolation based on domain decomposition in the next section.

4. Finite element extrapolations of differential operator eigenvalue problems

In Section 4.1, based on Theorem 3.1 and [22], the Richardson extrapolations of the nonconforming element for multiple eigenvalues are discussed. In Section 4.2, by using Theorem 3.1 and [27,28], the splitting extrapolations based on domain decomposition for multiple eigenvalues are studied. For a general theory of finite element methods we refer to [3,6,11,32].

4.1. The c f i gfi i e e e e e a a i f i e eige a e

Let $H^{\cdot}(\Omega)$ be a Sobolev space with norm $\|\cdot\|_{\ ,2}$, and $H_0^1(\Omega)$ be the subspace of $H^1(\Omega)$ consisting of those functions which vanish on $\partial\Omega$.

Consider the eigenvalue problem: Find a real number

Then the nonconforming EQ_1 element approximation corresponding to (4.1) is: Find a real number $\lambda_h \in \mathbf{R}$, $0 \neq \ _h \in S^h$, such that

$$a_h(\ _h, \ ) = \lambda_h b(\ _h, \ ), \quad \forall \ \in S^h. \tag{4.2}$$

Here, the eigenpair $(\lambda_h, \ _h)$ of (4.2) approximates the eigenpair $(\lambda, \ )$ of (4.1).

In order to discuss the error estimates, we will need the form of operator equation of (4.1) and (4.2). Define the operators $T, T_h : L_2(\Omega) \to L_2(\Omega)$:

$$a(Tf, \ ) = b(f, \ ), \quad \forall f \in L_2(\Omega), \ \forall \ \in H_0^1(\Omega),$$
$$a_h(T_h f, \ ) = b(f, \ ), \quad \forall f \in L_2(\Omega), \ \forall \ \in S^h.$$

Set $X = L_2(\Omega)$, $\langle \cdot,\cdot \rangle = b(\cdot,\cdot)$ and $\| \cdot \| = \| \cdot \|_b = \sqrt{b(\cdot,\cdot)}$. It can be easily seen that (4.1) and (4.2) have the equivalent operator forms (2.1) and (2.2), respectively, T and T_h are selfadjoint completely continuous operators (see for example [36]). By the EQ_1 element error estimate of the source problem corresponding to (4.1) (see [24,26]), we can deduce that $\|T - T_h\| \to 0$ $(h \to 0)$.

Let λ be the -th eigenvalue of (4.1) with algebraic multiplicity , $\lambda \ = \lambda_{\ +1} = \cdots = \lambda_{\ + \ -1}$, $\hat{\lambda}_{\ ,h} = (\frac{1}{\ } \sum_{j=\ }^{\ + \ -1} \lambda_{j,h}^{-1})^{-1}$. Based on [22] we obtain the following:

Theorem 4.1. *Le* $R(E) \subset H^5(\Omega)$. *The f he , , c , f i g* EQ_1 *e e e , he e h d he a ic e a , i , f he e f* $\hat{\lambda}_h$:

$$\hat{\lambda}_{\ ,h} - \lambda \ = -\frac{1}{\ } \sum_{j=\ }^{\ + \ -1} \frac{h_1^2 + h_2^2}{3} \int_\Omega \partial_1^2 \partial_2^2 \varphi_j \varphi_j + \vartheta(h^4), \tag{4.3}$$

he e $\partial_i = \frac{\partial}{\partial \ _i}$, $\partial_i^2 = \frac{\partial^2}{\partial \ _i^2}$, $i = 1, 2$.

Proof. Note that $\mu \ = \frac{1}{\lambda}$, $\hat{\mu}_{\ ,h} = \frac{1}{\hat{\lambda}_{\ ,h}}$, T and T_h are selfadjoint operators; from (3.1) and (3.2) we have

$$\frac{\hat{\lambda}_{\ ,h} - \lambda}{\lambda \ \hat{\lambda}_{\ ,h}} = \frac{1}{\ } \sum_{j=\ }^{\ + \ -1} b\big((T - T_h)\varphi_j, \varphi_j\big) + R,$$
$$|R| \leqslant C \big\| (T - T_h)|_{R(E)} \big\|^2. \tag{4.4}$$

From [3] and [22], we can see that

$$|\hat{\lambda}_{\ ,h} - \lambda \ | \leqslant Ch^2, \qquad \big\| (T - T_h)|_{R(E)} \big\| \leqslant Ch^2.$$

Then (4.4) can be written as

$$\hat{\lambda}_{\ ,h} - \lambda \ = \frac{1}{\ } \lambda^2 \sum_{j=\ }^{\ + \ -1} b\big((T - T_h)\varphi_j, \varphi_j\big) + \vartheta(h^4). \tag{4.5}$$

According to the error estimate of the nonconforming EQ_1 element, we have

$$\big| b(T\varphi_j - T_h\varphi_j, \varphi_j - \varphi_{j,h}) \big| \leqslant \|T\varphi_j - T_h\varphi_j\|_b \|\varphi_j - \varphi_{j,h}\|_b \leqslant Ch^4,$$

and

$$b(I_h T\varphi_j - T_h\varphi_j, \varphi_{j,h}) = \frac{1}{\lambda_j} a_h(I_h T\varphi_j - T_h\varphi_j, \varphi_{j,h}) + \vartheta(h^4),$$

where I_h is the interpolation operator of EQ_1 element. Thus

$$\begin{aligned} b\big((T - T_h)\varphi_j, \varphi_j\big) &= b(T\varphi_j - T_h\varphi_j, \varphi_j - \varphi_{j,h}) + b(T\varphi_j - I_h T\varphi_j, \varphi_{j,h}) + b(I_h T\varphi_j - T_h\varphi_j, \varphi_{j,h}) \\ &= \frac{1}{\lambda_j} b(\varphi_j - I_h\varphi_j, \varphi_{j,h}) + \frac{1}{\lambda_j} a_h(I_h T\varphi_j - T_h\varphi_j, \varphi_{j,h}) + \vartheta(h^4) \\ &= \frac{1}{\lambda_j} \big(b(\varphi_j - I_h\varphi_j, \varphi_{j,h}) + a_h(I_h T\varphi_j - T\varphi_j, \varphi_{j,h}) \\ &\quad + a_h(T\varphi_j - T_h\varphi_j, \varphi_{j,h}) \big) + \vartheta(h^4). \end{aligned} \tag{4.6}$$

From (3.1) in [22], we can see that

$$a_h(I_h T\varphi_j - T\varphi_j, \varphi_{j,h}) = 0.$$

From (3.3) in [22], it follows that

$$a_h(T\varphi_j - T_h\varphi_j, \varphi_{j,h}) = -\frac{h_1^2 + h_2^2}{3}\int_\Omega \partial_1^2\partial_2^2 T\varphi_j\varphi_{j,h} + \vartheta\left(h^4\right)\|T\varphi_j\|_5\|\varphi_{j,h}\|_h$$
$$= -\frac{h_1^2 + h_2^2}{3\lambda_j}\int_\Omega \partial_1^2\partial_2^2 \varphi_j\varphi_j + \vartheta\left(h^4\right)\|\varphi_j\|_5\|\varphi_{j,h}\|_h.$$

From (3.9) in [22], we can derive

$$b(\varphi_j - I_h\varphi_j, \varphi_{j,h}) = \vartheta\left(h^4\right).$$

Substituting the above three equalities into (4.6) we have

$$b\big((T - T_h)\varphi_j, \varphi_j\big) = -\frac{h_1^2 + h_2^2}{3\lambda_j^2}\int_\Omega \partial_1^2\partial_2^2\varphi_j\varphi_j + \vartheta\left(h^4\right).$$

Substituting the above equality into (4.5) we get (4.3). □

Algorithm 1 *(The Richa d e a ai f he c f i g e e e).*

Step 1. Under the partition with the mesh parameters h_1, h_2, solving (4.2) we can get $\hat{\lambda}_{\ ,h}$.
Step 2. Under the partition with the mesh parameters $\frac{h_1}{2}$, $\frac{h_2}{2}$, solving (4.2) we can get $\hat{\lambda}_{\ ,\frac{h}{2}}$.
Step 3. Compute the value of the extrapolation

$$\lambda^{e}_{\ ,h} = \frac{4}{3}\hat{\lambda}_{\ ,\frac{h}{2}} - \frac{1}{3}\hat{\lambda}_{\ ,h}.$$

Corollary 4.1. *U de he c di i f The e 4.1, he e h d*

$$\lambda^{e}_{\ ,h} - \lambda \ = \vartheta\left(h^4\right). \tag{4.7}$$

Proof. From (4.3), it can be seen that

$$\hat{\lambda}_{\ ,\frac{h}{2}} - \lambda \ = -\frac{1}{\ } \sum_{j=\ }^{\ +\ -1} \frac{1}{3}\left(\left(\frac{h_1}{2}\right)^2 + \left(\frac{h_2}{2}\right)^2\right)\int_\Omega \partial_1^2\partial_2^2\varphi_j\varphi_j + \vartheta\left(h^4\right).$$

Combining the above equality and (4.3), we obtain (4.7). □

Eq. (4.7) indicates that the nonconforming EQ_1 element extrapolations for multiple eigenvalues achieve the accuracy order $\vartheta(h^4)$.

Remark 4.1. Thanks to [22], we can also study multiple eigenvalues extrapolations of the nonconforming Q o81 5 ()Tj /T1-467(e)2448 where

$$a(\ ,\ ) = \int_{\Omega} \left(\sum_{i,j=1}^{d} a_{ij}(\ )\partial_i\ \overline{\partial_j\ } + \sum_{i=1}^{d} b_i(\ )\partial_i\ \bar{\ } + c(\ )\ \bar{\ } \right) d\ ,$$

$$b(\ ,\ ) = \int_{\Omega} \rho(\ )\ \bar{\ } d\ .$$

Assume that $a_{ij}(\ ), c(\ ), \rho(\ ) \in L_\infty(\Omega)$, $b_i(\ ) \in W_{1,\infty}(\Omega)$, $\ = (\ _1, \ldots, \ _d)$, and there exists a positive constant a_0 such that

$$Re \sum_{i,j=1}^{d} a_{ij}(\ )\xi_i\xi_j \geqslant a_0 \sum_{i=1}^{d} \xi_i^2, \quad \forall\ \in \Omega,\ \forall(\xi_1, \ldots, \xi_d)^T \in R^d,$$

$$Re\, c(\ ) \geqslant \frac{1}{2}a_0 + \frac{1}{2} \max_{\ \in\Omega,\, i=1,2} \left|b_i(\ )\right|^2 / a_0, \quad \forall\ \in \Omega.$$

Let G be a convex quadrilateral. A quadrilateral grid by connecting the equidistant mesh points of the opposite edges of G is called a strong regular division of G. Let G be a convex hexahedral. A three-dimensional partition by connecting the corresponding strong regular division nodes of the opposite faces is called a strong regular division of G. For two-dimensional case in strong regular division there are two independent grid parameters which are determined by the numbers of the equidistant mesh points of the opposite edges. For three-dimensional case there are three independent grid parameters in strong regular division.

Let $\overline{\Omega} = \bigcup_{\ =1} \overline{\Omega}_{\ }$, where $\Omega_{\ }$ is a convex quadrilateral ($d = 2$) or convex hexahedron ($d = 3$), and the initial partition satisfies the compatible condition and has no interior cross points. Then for each $\Omega_{\ }$, we construct such a strong regular partition $\tau_{\ ,h}$ that $\tau_h = \bigcup_{\ =1} \tau_{\ ,h}$ is a piecewise strongly regular partition of the domain Ω.

Note that the partition $\tau_{\ ,h}$ has d mesh parameters $h_{\ }$ ($\ = 1, \ldots, d$) while the partition τ_h only has $\ $ ($\ < \ \times d$) independent mesh parameters denoted by $\bar{h}_1, \ldots, \bar{h}_{\ }$. Let $h_0 = \max_{1 \leqslant i \leqslant\ } \bar{h}_i$.

When $\Omega_{\ }$ is a quadrilateral, $h_{\ }$ is the side-length of an element on $\partial\Omega_{\ }$. When $\Omega_{\ }$ is a hexahedron, $h_{\ }$ is the edge-length of an element on $\partial\Omega_{\ }$. Since τ_h is a piecewise strongly regular partition, there exist constants C_1 and C_2 independent of the mesh diameter h such that

$$C_1 h \leqslant h_0 \leqslant C_2 h.$$

orthonormal basis for $R(E)$ and let $\varphi_j^* = E^*\varphi_j$ (see [3, p. 691]). From [3], we can see that (4.8) and (4.9) have the equivalent operator forms (2.1) and (2.2), respectively, $\|T - T_h\| \to 0$ $(h \to 0)$ and (4.10) is equivalent to the operator form $T^* \quad = \mu^* \quad$.

Let λ be the -th eigenvalue of (4.8) with algebraic multiplicity , $\lambda \ = \lambda_{\ +1} = \cdots = \lambda_{\ + \ -1}$, $\hat{\lambda}_{\ ,h} = (\frac{1}{\ } \sum_{j=}^{\ + \ -1} \lambda_{j,h}^{-1})^{-1}$. Based on [27] we obtain the following:

Theorem 4.2. *S , e ha he c efficie , f diffe e ia e a a e iece i e , h:* $a_{ij} \in (\prod_{\ =1} C^5(\Omega \ )) \cap L_\infty(\Omega)$, $b_i \in (\prod_{\ =1} C^5(\Omega \ )) \cap W_{1,\infty}(\Omega)$, $c \in (\prod_{\ =1} C^4(\Omega \ )) \cap L_\infty(\Omega)$, *a d* $\rho \in (\prod_{\ =1} C^4(\Omega \ )) \cap L_\infty(\Omega)$. *A , e ha* $R(E) \subset (\prod_{\ =1} H^6(\Omega \ )) \cap H_0^1(\Omega)$. *The he e h d he a ic e a , i , f he e f* $\hat{\lambda}_{\ ,h}$:

$$\hat{\lambda}_{\ ,h} - \lambda \ = \sum_{i=1} \beta_i \bar{h}_i^2 + \vartheta\left(h_0^4\right),$$

From (26) in [27] we have

$$a\big(T\varphi_j - I_hT\varphi_j, T^*\varphi_j^*\big) = b\big(T\varphi_j - I_hT\varphi_j, \varphi_j^*\big)$$
$$= \sum_{=1} \sum_{=1}^{d} \frac{1}{12} h^2 \int_{\Omega} \rho(\ )\varphi_j^* \partial^2 T\varphi_j \, d \ + \vartheta\big(h_0^4\big).$$

Substituting the above two equalities into (4.16), together with (4.15)

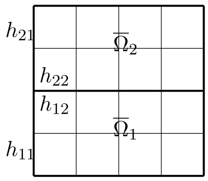

Fig. 1. $(\bar{h}_1, \bar{h}_2, \bar{h}_3)$.

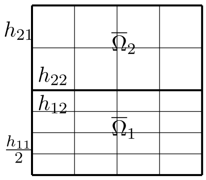

Fig. 2. $(\frac{\bar{h}_1}{2}, \bar{h}_2, \bar{h}_3)$.

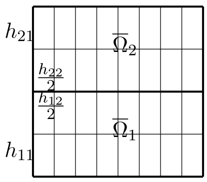

Fig. 3. $(\bar{h}_1, \frac{\bar{h}_2}{2}, \bar{h}_3)$.

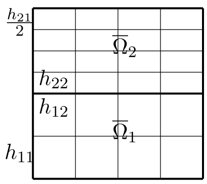

Fig. 4. $(\bar{h}_1, \bar{h}_2, \frac{\bar{h}_3}{2})$.

Example 5.2. Consider the non-symmetric convection diffusion problem:

$$-\triangle \quad + b \cdot \triangledown \quad = \lambda \quad , \quad \text{in } \Omega, \qquad = 0, \quad \text{on } \partial\Omega, \tag{5.1}$$

where $b = (1, 1)$, and Ω is a square $(0, 1) \times (0, 1)$.

We know that the multiplicities of the eigenvalues $\lambda_2 = \lambda_3$ and $\lambda_5 = \lambda_6$ are all equal to 2, and

$$\lambda_2 = \lambda_3 \approx 49.848022005, \qquad \lambda_5 = \lambda_6 \approx 99.196044011.$$

We make a uniform square partition for Ω and use the conforming bilinear element to compute the approximate eigenvalues corresponding to (5.1). We decompose $\overline{\Omega}$ as: $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, where $\overline{\Omega}_1 = [0, 1] \times [0, \frac{1}{2}]$ and $\overline{\Omega}_2 = [0, 1] \times [\frac{1}{2}, 1]$. Let $\tau_{\ ,h}$ be the partition on $\overline{\Omega}_{\ }$ $(\ = 1, 2)$.

Note that $\tau_{1,h}$ has 2 mesh parameters h_{11} and h_{12}, $\tau_{2,h}$ has 2 mesh parameters h_{21} and h_{22}, but τ_h only has 3 independent mesh parameters $\bar{h}_1 = h_{11}$, $\bar{h}_2 = h_{12} = h_{22}$ and $\bar{h}_3 = h_{21}$ (see Figs. 1–4). Let $h_0 = \max_{1 \leqslant i \leqslant 3} \bar{h}_i$.

Let $\lambda_{\ ,h}$, $\lambda^{(1)}_{\ ,h}$, $\lambda^{(2)}_{\ ,h}$ and $\lambda^{(3)}_{\ ,h}$ be the approximate eigenvalues corresponding to the bilinear finite element approximations on the partition with mesh parameters $(\bar{h}_1, \bar{h}_2, \bar{h}_3)$, $(\frac{\bar{h}_1}{2}, \bar{h}_2, \bar{h}_3)$, $(\bar{h}_1, \frac{\bar{h}_2}{2}, \bar{h}_3)$ and $(\bar{h}_1, \bar{h}_2, \frac{\bar{h}_3}{2})$, respectively. Let $\hat{\lambda}_{\ ,h} = (\frac{1}{2}(\frac{1}{\lambda_{\ ,h}} + \frac{1}{\lambda_{\ +1,h}}))^{-1}$ $(\ = 2, 5)$, $\hat{\lambda}^{(i)}_{\ ,h} = (\frac{1}{2}(\frac{1}{\lambda^{(i)}_{\ ,h}} + \frac{1}{\lambda^{(i)}_{\ +1,h}}))^{-1}$ $(i = 1, 2, 3, \ \ = 2, 5)$. We denote the approximate eigenvalues obtained by Algorithm 2 by

$$\hat{\lambda}^{e}_{\ ,h} \equiv \frac{4}{3}\hat{\lambda}_{\ ,\frac{h}{2}} - \frac{1}{3}\hat{\lambda}_{\ ,h} \quad (\ = 2, 5),$$

Table 2
The numerical eigenvalues $\hat{\lambda}^{\ e}_{\ ,h}$ obtained by Algorithm 2, and $\hat{\lambda}^{\ e}_{\ ,h}$ obtained by Algorithm 3 ($\ = 2, 5$).

h_0	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
$\hat{\lambda}_{2,h}$	51.951541	50.365882	49.976951	49.880220	49.856069
$\hat{\lambda}^{(1)}_{2,h}$, $\hat{\lambda}^{(3)}_{2,h}$	51.544287	50.267955	49.952725	49.874180	49.854560
$\hat{\lambda}^{(2)}_{2,h}$	51.148823	50.170812	49.928548	49.868143	49.853051
$\hat{\lambda}^{e}_{2,h}$	49.837329	49.847307	49.847976	49.848019	
$\hat{\lambda}^{e}_{2,h}$	49.795240	49.844650	49.847811	49.848011	
$\hat{\lambda}_{5,h}$	109.81232	101.77789	99.835956	99.355663	99.235926
$\hat{\lambda}^{(1)}_{5,h}$, $\hat{\lambda}^{(3)}_{5,h}$	107.57658	101.27850	99.715022	99.325675	99.228444
$\hat{\lambda}^{(2)}_{5,h}$	105.64870	100.79796	99.595254	99.295760	99.220967
$\hat{\lambda}^{e}_{5,h}$	99.099747	99.188645	99.195565	99.196014	
$\hat{\lambda}^{e}_{5,h}$	98.298853	99.139610	99.192529	99.195824	

and the approximate eigenvalues by Algorithm 3 by

$$\hat{\lambda}^{\ e}_{\ ,h} \equiv \sum_{j=1}^{3} \frac{4}{3}\hat{\lambda}^{(j)}_{\ ,h} - \left(\frac{4}{3} \times 3 - 1\right)\hat{\lambda}_{\ ,h} \quad (\ = 2, 5).$$

Numerical results listed in Table 2 are in accordance with our theoretical analysis.

Remark 5.1. We can see from Example 5.2 that the numerical eigenvalue by the splitting extrapolation has the same accuracy with that by the Richardson extrapolation. The splitting extrapolation method needs to solve four subproblems. The number of nodes of the maximum subproblem, which is illustrated in Fig. 3, are just a half of the number of nodes on the fine grid of the Richardson extrapolation method. When considering Example 5.2 in $\Omega \subset R^3$, we make a split $\Omega = \Omega_1 \cup \Omega_2$. From [27] we know that, to obtain the same accuracy, the splitting extrapolation method needs to solve five subproblems, and the number of nodes of the maximum subproblem is just a quarter of the number of nodes on the fine grid of the Richardson extrapolation method. The higher the dimensions are, the more superior the splitting extrapolation method performs.

6. Concluding remarks

This paper discusses the extrapolation of numerical eigenvalues by finite elements for differential operators. Theorem 3.1 in the paper provides a new error expansion. Using this error expansion we solve the extrapolation for multiple eigenvalues, which was once thought to be a complicated work. For instance, in the paper we achieve the extrapolation of nonconforming finite elements for multiple eigenvalues and the splitting extrapolation based on domain decomposition of conforming finite elements for multiple eigenvalues (including the case that the ascent is larger than 1) of non-selfadjoint differential operator. Although the proof of Theorem 3.1 is just a minor modification of that of Osborn (see Theorem 3 of [29], Theorem 7.2 of [3]), Theorem 3.1 develops the spectral approximation theory and is a general result.

Acknowledgements

The authors cordially thank Professor Claude Brezinski and Professor Zi Cai Li for their valuable comments and suggestions that led to the improvement of this paper.

References

[1] M. Asadzadeh, A.H. Schatz, W. Wendland, A new approach to Richardson extrapolation in the finite element method for second order elliptic problems, Math. Comput. 78 (2009) 1951–1973.
[2] M. Asadzadeh, A.H. Schatz, W. Wendland, Asymptotic error expansions for the finite element method for second order elliptic problems in R^N, $N \geqslant 2$, I: local interior expansions, SIAM J. Numer. Anal. 48 (5) (2009) 2000–2017.
[3] I. Babuska, J. Osborn, Eigenvalue problems, in: P.G. Ciarlet, J.L. Lious (Eds.), Handbook of Numerical Analysis, vol. II, Finite Element Method (Part I), Elsevier Science Publishers B.V., North-Holand, 1991, pp. 640–787.
[4] H. Blum, R. Rannacher, Extrapolation techniques for reducing the pollution effect or reentrant corners in the finite element method, Numer. Math. 52 (5) (1988) 539–564.
[5] H. Blum, R. Rannacher, Finite element eigenvalue computation on domains with reentrant corners using Richardson extrapolation, J. Comput. Math. 8 (3) (1990) 321–332.
[6] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, second ed., Springer-Verlag, New York, 2002.
[7] Y. Cao, X. He, T. Lü, A splitting extrapolation for solving nonlinear elliptic equations with d-quadratic finite elements, J. Comput. Phys. 228 (2009) 109–122.
[8] F. Chatelin, Spectral Approximations of Linear Operators, Academic Press, New York, 1983.
[9] C.M. Chen, Y.Q. Huang, High Accuracy Theory of Finite Element Methods, Hunan Science & Technology Publisher, Changsha, 1995 (in Chinese).

[10] H.S. Chen, B. Li, Superconvergence analysis and error expansion for the Wilson nonconforming finite element, Numer. Math. 69 (1994) 125–140.

[11] P.G. Ciarlet, Basic error estimates for elliptic problems, in: P.G. Ciarlet, J.L. Lious (Eds.), Handbook of Numerical Analysis, vol. II, Finite Element Method (Part I), Elsevier Science Publishers B.V., North-Holand, 1991, pp. 21–343.

[12] N. Dunford, J.T. Schwartz, Liner Operators, vol. 2: Spectral Theory, Selfadjoint Operators in Hilbert Space, Interscience, New York, 1963.

[13] G. Fairweather, Q. Lin, Y.P. Lin, J.P. Wang, S.H. Zhang, Asymptotic expansions and Richardson extrapolation of approximate solutions for second order elliptic problems on rectangular domains by mixed finite element methods, SIAM J. Numer. Anal. 44 (2006) 1122–1149.

[14] X. He, T. Lü, A finite element splitting extrapolation for second order hyperbolic equations, SIAM J. Sci. Comput. 31 (6) (2009) 4244–4265.

[15] S. Jia, H. Xie, X. Yin, S. Gao, Approximation and eigenvalue extrapolation of biharmonic eigenvalue problem by nonconforming finite element methods, Numer. Methods Partial Differential Equations 24 (2008) 435–448.

[16] D.C. Joyce, Survey of extrapolation processes in numerical analysis, SIAM Rev. 13 (1971) 435–490.

[17] Z.C. Li, H.T. Huang, N. Yan, Global Superconvergence of Finite Elements for Elliptic Equations and Its Applications, Science Press, Beijing, 2012.

[18] M. Li, Q. Lin, S. Zhang, Extrapolation and superconvergence of the Steklov eigenvalue problem, Adv. Comput. Math. 33 (2010) 25–44.

[19] X.H. Liao, A.H. Zhou, A multi-parameter splitting extrapolation and a parallel algorithm for elliptic eigenvalue problem, J. Comput. Math. 16 (3) (1998) 213–220.

[20] C.B. Liem, T. Lü, T.M. Shin, The Splitting Extrapolation Method, a New Technique in Numerical Solution of Multidimensional Problems, with a preface by Zhong-ci Shi, Series on Applied Mathematics, World Scientific, Singapore, 1995.

[21] Q. Lin, Fourth order eigenvalue approximation by extrapolation on domains with reentrant corners, Numer. Math. 58 (1991) 631–640.

[22] Q. Lin, H.T. Huang, Z.C. Li, New expansions of numerical eigenvalues for $-\Delta \quad = \lambda\rho$ by nonconforming elements, Math. Comput. 77 (264) (2008) 2061–2084.

[23] Q. Lin, H.T. Huang, Z.C. Li, New expansions of numerical eigenvalues by Wilson's elements, J. Comput. Appl. Math. 225 (2009) 213–226.

[24] Q. Lin, J. Lin, Finite Element Methods: Accuracy and Improvement, Science Press, Beijing, 2006.